# A simple explicit homogenization solution for the macroscopic response of isotropic porous elastomers

Bhavesh Shrimali[1], Victor Lefèvre[2], Oscar Lopez-Pamies[1]

[1] Department of Civil and Environmental Engineering,
University of Illinois at Urbana-Champaign

[2] Department of Mechanical Engineering,
Northwestern University

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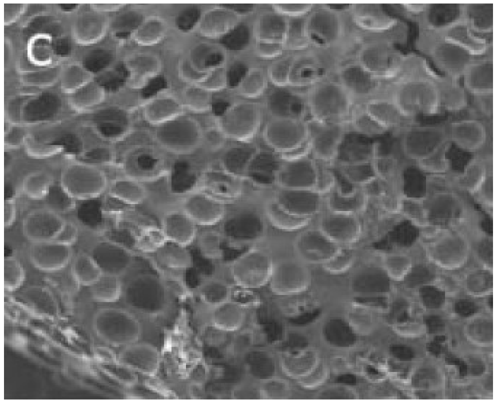
# Motivation

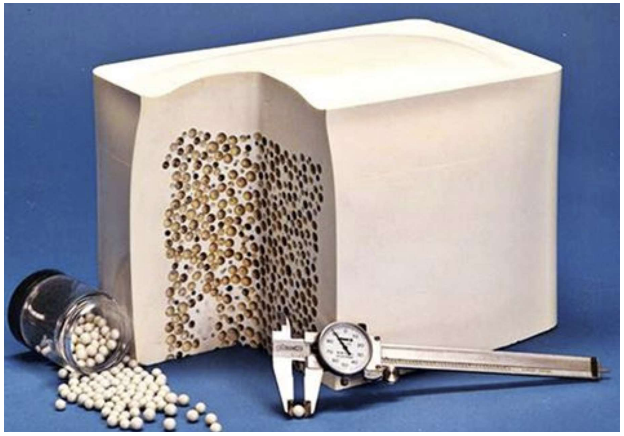

Polyurethane foam

Closed-cell neoprene




SIS block copolymer foam

SBR Foam



Syntactic foam

# Kinematics \ Local Constitutive Behavior

$$W_{\mathtt{m}}(\mathbf{QFK}) = W_{\mathtt{m}}(\mathbf{F}) \quad \forall \quad \mathbf{Q},\mathbf{K} \in Orth^{+}$$

$$W_{\mathtt{m}}(\mathbf{F}) = \begin{cases} \Psi_{\mathtt{m}}(I_1) & \text{if} \quad J = 1 \\ +\infty & \text{otherwise} \end{cases}$$

$$I_1 = \mathbf{F}\cdot\mathbf{F} \qquad J = \det \mathbf{F}$$

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{X},\mathbf{F})\,;$$

$$W(\mathbf{X},\mathbf{F}) = [1-\theta_0(\mathbf{X})]\,W_{\mathtt{m}}(\mathbf{F})$$

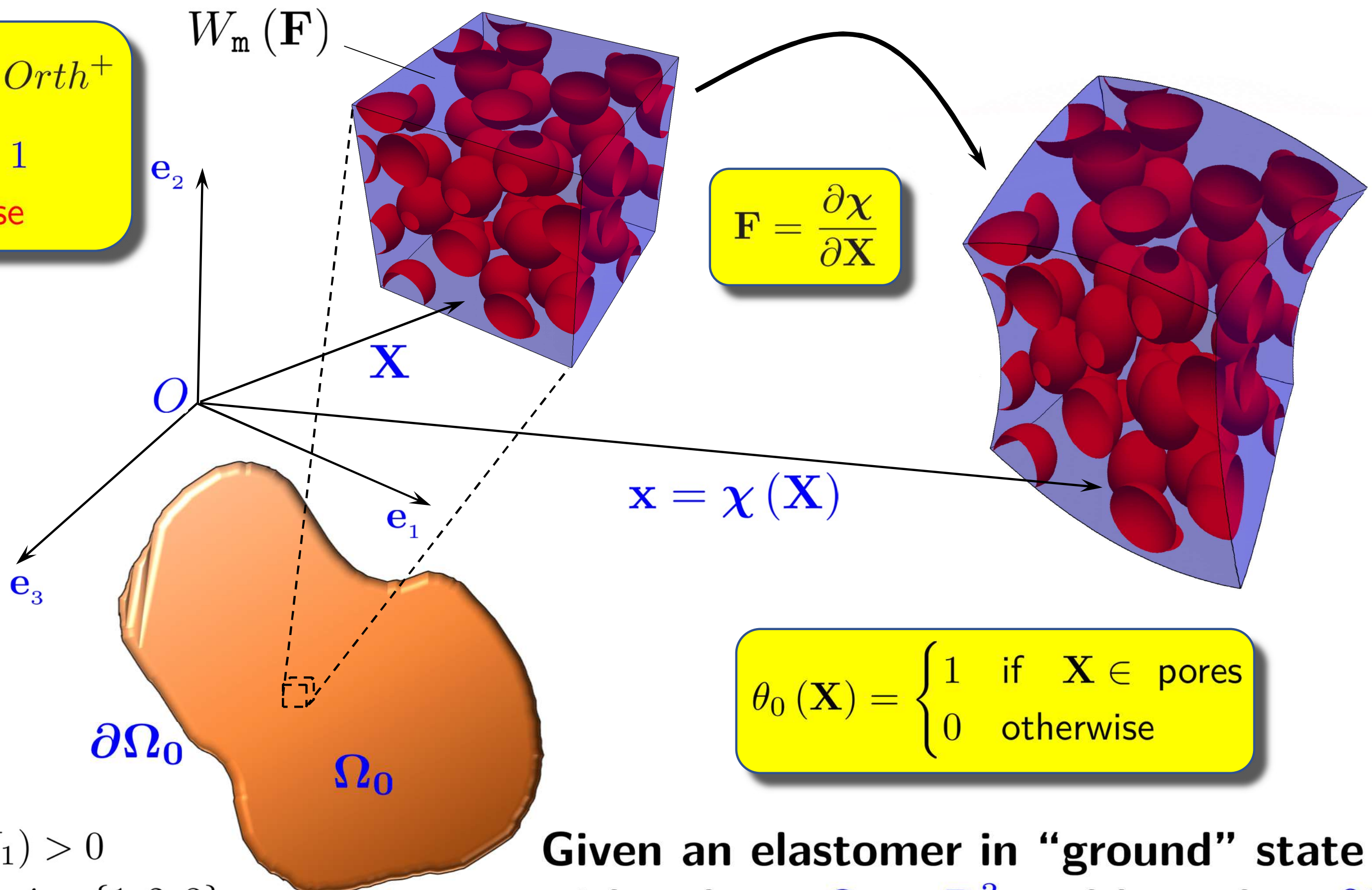
$\Psi_{\mathtt{m}}(3) = 0$ and $\Psi'_{\mathtt{m}}(3) = \dfrac{\mu}{2}$

$\Psi'_{\mathtt{m}}(I_1) > 0;$

$$\Psi'_{\mathtt{m}}(I_1) + 2\left(I_1 - \lambda_i^2 - \frac{2}{\lambda_i}\right)\Psi''_{\mathtt{m}}(I_1) > 0$$

$$i = \{1,2,3\}$$

Zee and Sternberg, Arch. Rat. Mech. Anal. *(1983)*




$$\mathbf{F} = \frac{\partial \chi}{\partial \mathbf{X}}$$

$$\theta_0(\mathbf{X}) = \begin{cases} 1 & \text{if} \quad \mathbf{X} \in \text{pores} \\ 0 & \text{otherwise} \end{cases}$$

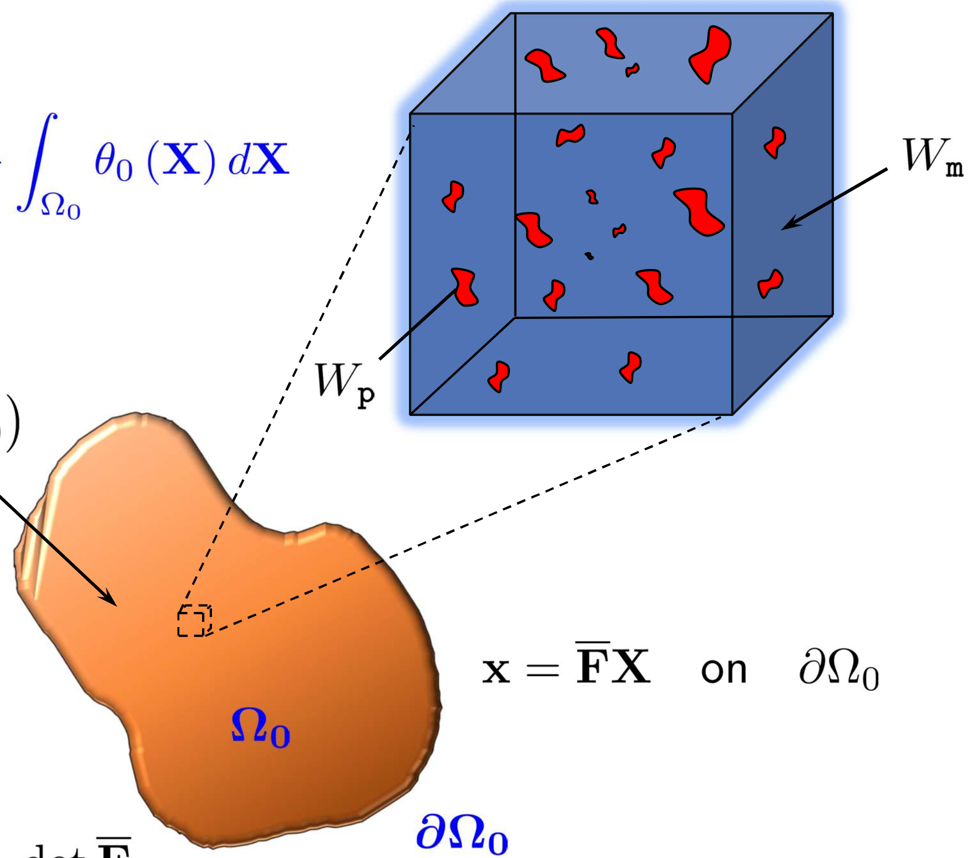
**Given an elastomer in "ground" state with volume $\Omega_0 \subset \mathbb{R}^3$ and boundary $\partial\Omega_0$**

# Macroscopic description of Elastomers

$$\overline{\mathbf{S}} = \frac{\partial \overline{W}}{\partial \overline{\mathbf{F}}}(\overline{\mathbf{F}}, f_0),$$

where $\overline{W}(\overline{\mathbf{F}}, f_0) \doteq \min_{\mathbf{F}\in\mathcal{K}} \int_{\Omega_0} W(\mathbf{X},\mathbf{F})\mathrm{d}\mathbf{X},$

$$f_0 \doteq \frac{1}{|\Omega_0|}\int_{\Omega_0} \theta_0(\mathbf{X})\, d\mathbf{X}$$




$$\mathbf{Div}\,\mathbf{S} = \mathbf{0} \quad \mathbf{X} \in \Omega_0$$

$$\mathbf{x} = \overline{\mathbf{F}}\mathbf{X} \quad \text{on} \quad \partial\Omega_0$$

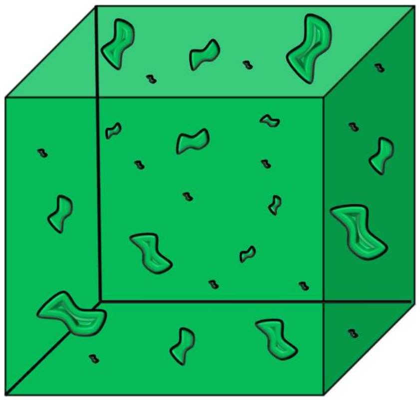
- $\overline{W}$ **is an isotropic function of** $\overline{\mathbf{F}}$
- $\overline{W}(\overline{\mathbf{F}}, f_0) = \overline{U}(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, f_0) = \overline{\Psi}(\overline{I}_1, \overline{I}_2, \overline{J}, f_0)$

$$\overline{I}_1 = \overline{\mathbf{F}} \cdot \overline{\mathbf{F}}, \quad \overline{I}_2 = \frac{1}{2}\left[\overline{I}_1^2 - (\overline{\mathbf{F}}^T\overline{\mathbf{F}}) \cdot (\overline{\mathbf{F}}^T\overline{\mathbf{F}})\right], \quad \overline{J} = \det \overline{\mathbf{F}}.$$

HILL, PROC. ROY. SOC. A. *(1972)*

# Main Result

- **For any type of non-percolative isotropic distribution of equi-axed closed-cell vacuous pores embedded in a Gaussian or Non-Gaussian elastomeric matrix**



$$\overline{W}(\overline{\mathbf{F}}, f_0) = (1 - f_0)\,\Psi_{\mathtt{m}}\left(\frac{\mathcal{I}_1}{1 - f_0} + 3\right)$$

$$\mathcal{I}_1 = \frac{3(1-f_0)}{3+2f_0}\left[\overline{I}_1 - 3\right] + \frac{3}{\overline{J}^{1/3}}\left[2\overline{J} - 1 - \frac{(1-f_0)\,\overline{J}^{1/3}\left(3\overline{J}^{2/3} + 2f_0\right)}{3+2f_0} - \frac{f_0^{1/3}\overline{J}^{1/3}\left(2\overline{J} + f_0 - 2\right)}{\left(\overline{J} - 1 + f_0\right)^{1/3}}\right]$$

- $\Psi_{\mathtt{m}}\left(\overline{\mathbf{F}}\right)$: **Stored-energy function (finite branch) for the elastomeric matrix**

$$\overline{I}_1 = \overline{\mathbf{F}}\cdot\overline{\mathbf{F}}, \quad \overline{I}_2 = \frac{1}{2}\left[\overline{I}_1^2 - (\overline{\mathbf{F}}^T\overline{\mathbf{F}})\cdot(\overline{\mathbf{F}}^T\overline{\mathbf{F}})\right], \quad \overline{J} = \det\overline{\mathbf{F}}.$$

Shrimali, Lefèvre, Lopez-Pamies, J. Mech. Phys. Sol. *(2019)*

# Porous Neo-Hookean elastomers: Explicit result

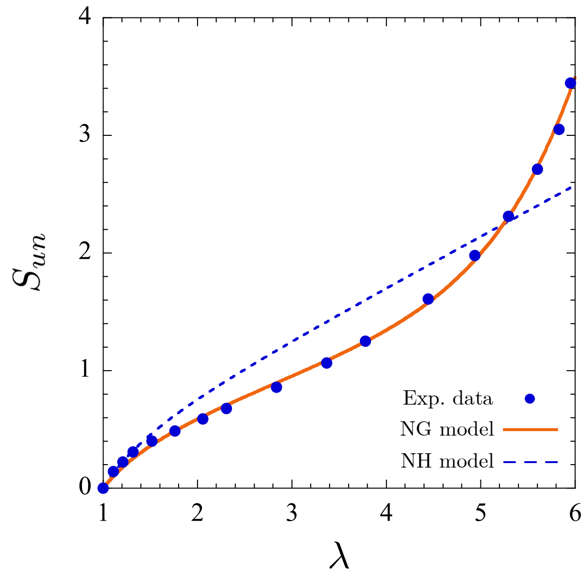
- **For the specific case when the underlying matrix is Neo-Hookean i.e.**

$$\Psi_{\mathtt{m}}(I_1) = \frac{\mu}{2}[I_1 - 3]$$

**the solution is given by**

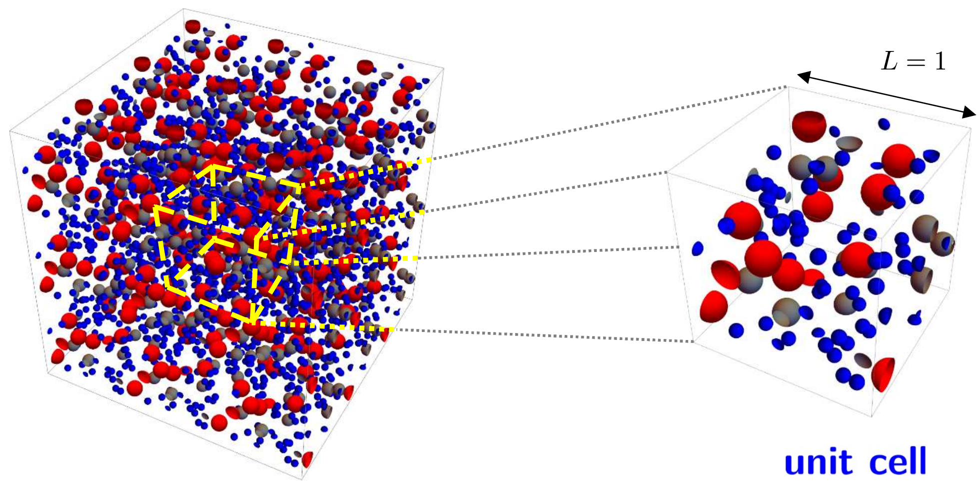
- **and is of the form**

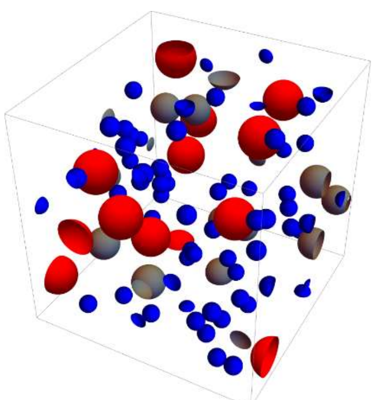
$$\overline{W} = \overline{W}_{\overline{I}_1}\left(\overline{I}_1, f_0\right) + \overline{W}_{\overline{J}}\left(\overline{J}, f_0\right)$$



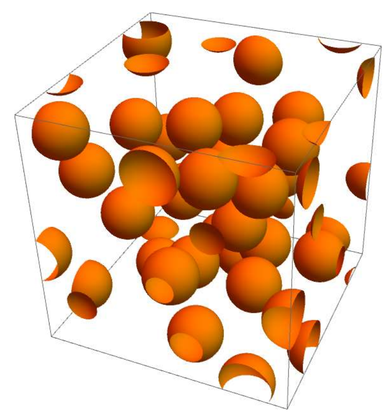

$$\overline{W}(\overline{\mathbf{F}}, f_0) = \frac{3(1-f_0)\mu_{\mathtt{m}}}{2\,(3+2f_0)}\left[\overline{I}_1 - 3\right] + \frac{3\mu_{\mathtt{m}}}{2\overline{J}^{1/3}}\left[2\overline{J} - 1 - \frac{(1-f_0)\,\overline{J}^{1/3}\left(3\overline{J}^{2/3} + 2f_0\right)}{3+2f_0} - \frac{f_0^{1/3}\overline{J}^{1/3}\left(2\overline{J} + f_0 - 2\right)}{\left(\overline{J} - 1 + f_0\right)^{1/3}}\right]$$

# Numerical homogenization: periodic microstructures

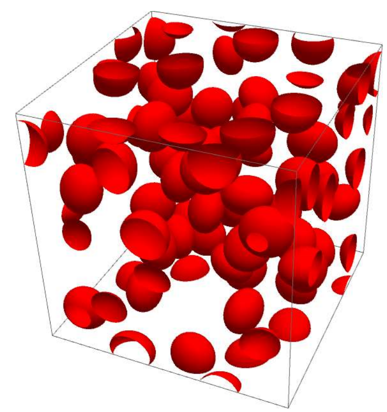


Finitely polydisperse spherical pores



Monodisperse spherical pores



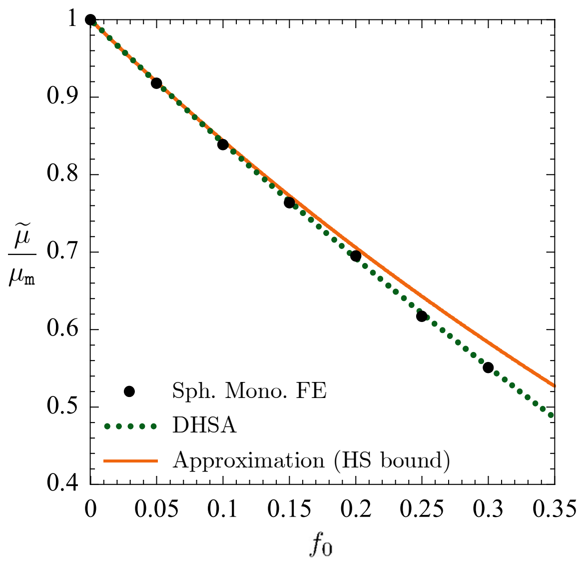
Monodisperse oblate spheroidal pores

# Numerical homogenization: small deformation limit

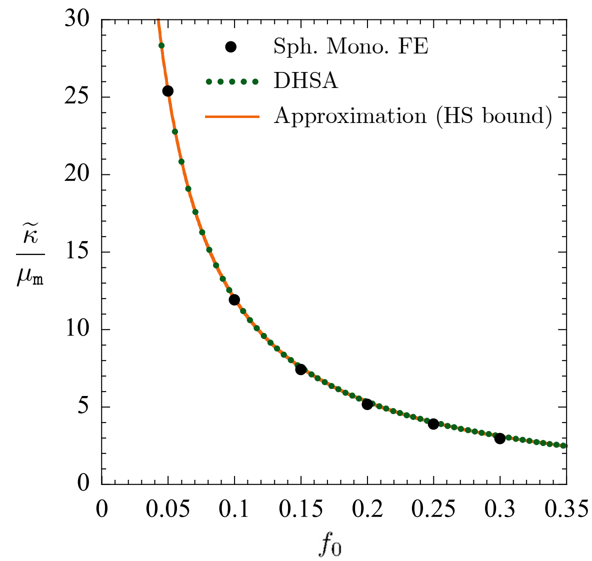
**Limit of small macroscopic deformations ($\overline{\mathbf{F}} \to \mathbf{I}$)**

$$\overline{W}(\overline{\mathbf{F}}, f_0) = \frac{\widetilde{\mu}}{2}\left[\bar{I}_1 - 3\right] - \widetilde{\mu}\left(\bar{J} - 1\right) + \frac{1}{2}\left(\widetilde{\kappa} + \frac{\widetilde{\mu}}{3}\right)\left(\bar{J} - 1\right)^2 + \mathcal{O}\left(\|\overline{\mathbf{F}} - \mathbf{I}\|^3\right),$$

$$\widetilde{\mu} = \frac{3(1 - f_0)}{3 + 2f_0}\mu$$




Effective initial shear modulus



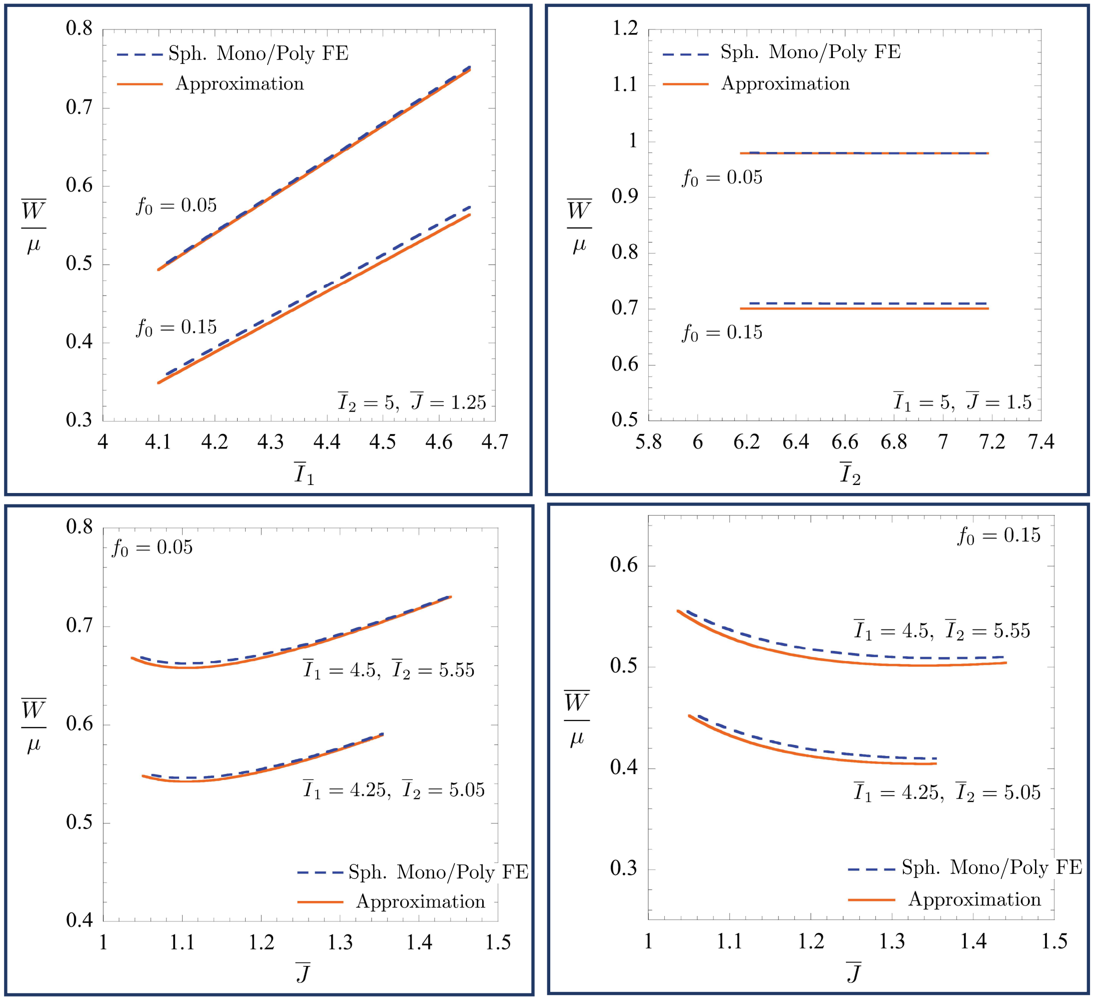

Effective initial bulk modulus

$$\widetilde{\kappa} = \frac{4(1 - f_0)}{3f_0}\mu.$$

# Key features

- The solution is roughly linear in $\overline{I}_1(\overline{\mathbf{F}})$
- The solution is independent of $\overline{I}_2(\overline{\mathbf{F}})$
- The solution is nonlinear in $\overline{J}(\overline{\mathbf{F}})$
- Of the form

$$\overline{W} = \overline{W}_{\overline{I}_1}\left(\overline{I}_1, f_0\right) + \overline{W}_{\overline{J}}\left(\overline{J}, f_0\right)$$



# Porous Neo-Hookean elastomers: Asymptotic limits

**Isotropic deformations ($\overline{\mathbf{F}} = \overline{J}^{1/3}\mathbf{I}$)**

$$\overline{W}\left(\overline{J}^{1/3}\mathbf{I}, f_0\right) = \frac{3\mu}{2}\left[\frac{2\overline{J}-1}{\overline{J}^{1/3}} - \frac{2\overline{J}+f_0-2}{(\overline{J}+f_0-1)^{1/3}} f_0^{1/3} - (1-f_0)\right]$$

- In spite of corresponding to a solution for a different type of microstructure, the above result agrees identially with the result of Hashin for a porous Neo-Hookean elastomer with the HSA microstructure under hydrostatic loading

**Limit of of infinitely large tensile deformations $\|\overline{\mathbf{F}}\| \to +\infty$**

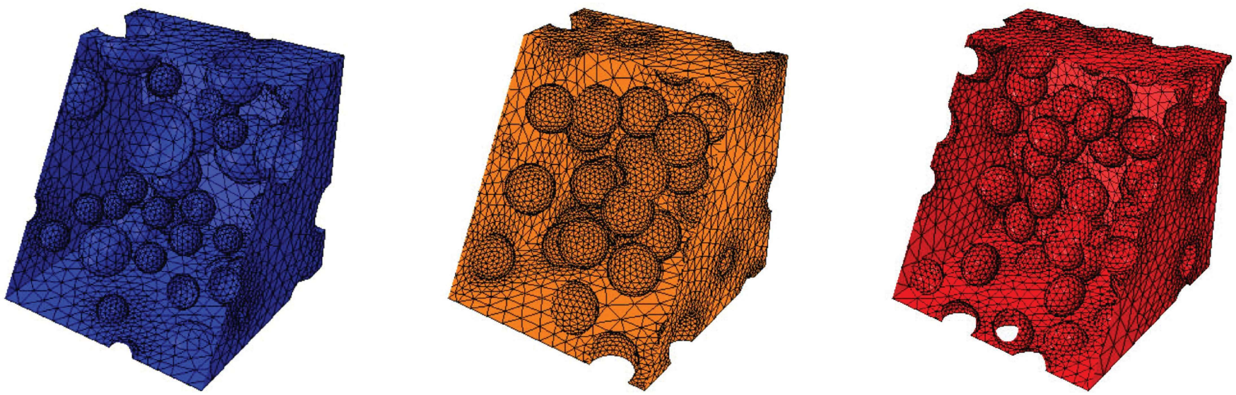
$$\overline{W}(\overline{\mathbf{F}}, f_0) = \frac{(1-f_0)\mu}{2+f_0}\,\overline{I}_1 + \frac{3(1+f_0^{1/3})(1-f_0^{1/3})^3\mu}{2+f_0}\,\overline{J}^{2/3}.$$

# Numerical homogenization: periodic microstructures...

- **Hybrid formulation in order to account for the (near) incompressibility of elastomeric matrix**

$$\boxed{\overline{W}\left(\overline{\mathbf{F}}, f_0\right) = \min_{\mathbf{u}\in\mathcal{U}} \max_{p\in\mathcal{P}} \int_Y \left\{ p[\det \mathbf{F}(\mathbf{u}) - 1] - \widehat{W}^*(\mathbf{X}, \mathbf{F}(\mathbf{u}), p) \right\} \mathrm{d}\mathbf{X}}$$

- **Conforming hybrid quadratic finite element discretizations**



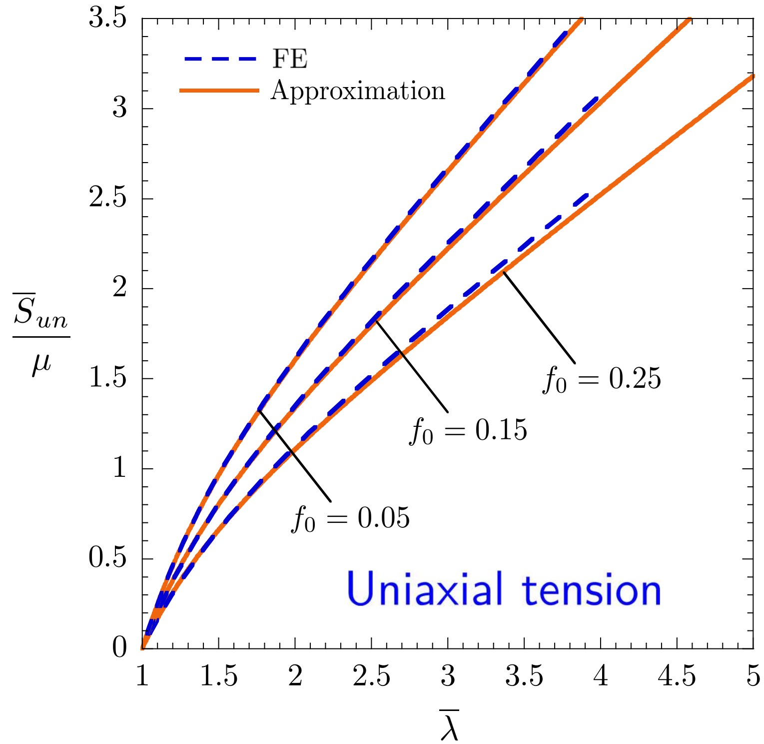
$N_{\mathrm{p}} = 5\,(N = 40)$
polydisperse spherical pores

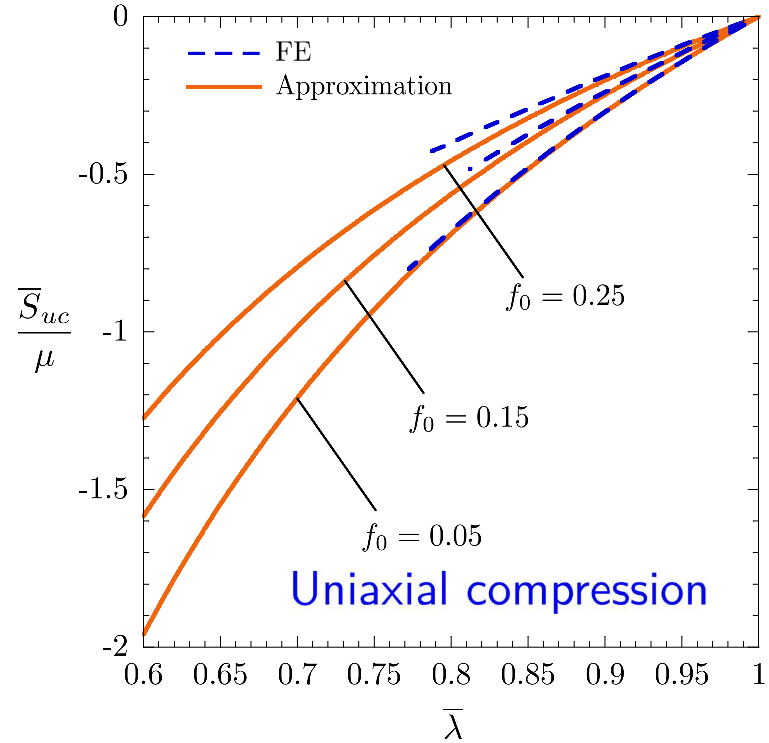
$(N = 30)$
monodisperse spherical pores

$(N = 60)$
monodisperse (oblate) spheroidal pores

# Numerical Solutions for arbitrary finite deformations

- **Lefèvre** recently put forth WENO solutions
- Numerical calculations using **FE calculations**

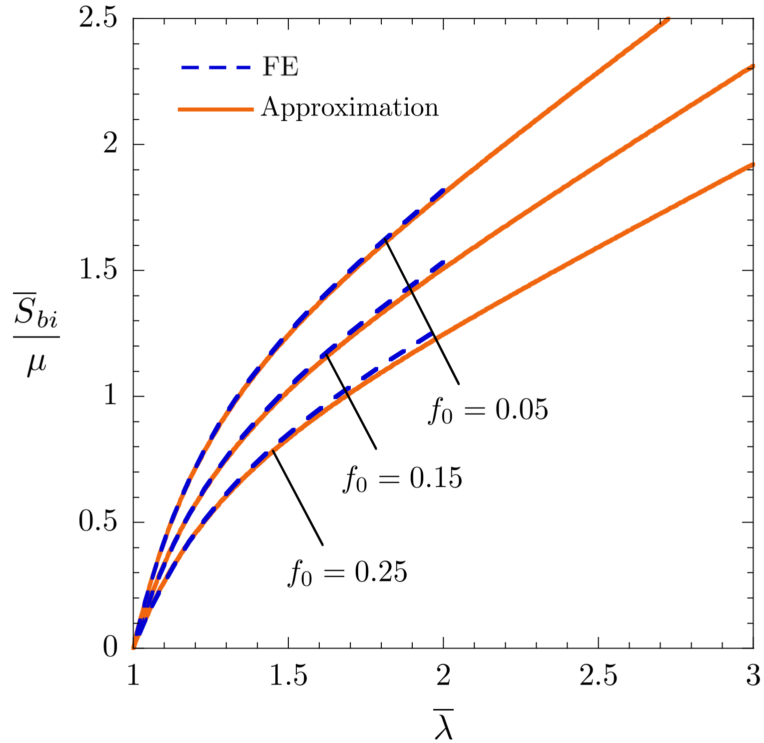


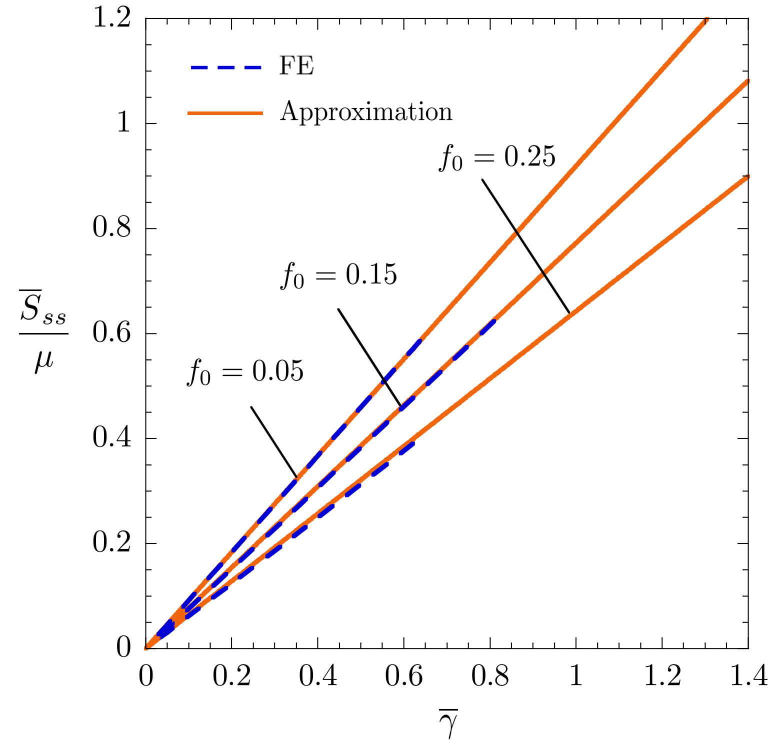

Lefèvre et al., Comp. Meth. Appl. Mech. Engg. *(2019)*

# Numerical Solutions for arbitrary finite deformations...

Biaxial tension




Simple Shear



# From Neo-Hookean to Non-Gaussian...

**Porous nonlinear comparison medium with the same microstructure as the actual material**

$$W_0(\mathbf{F},\mathbf{X}) = [1-\theta_0(\mathbf{X})]V_{\mathtt{m}}(\mathbf{F}) \qquad \text{with} \qquad V_{\mathtt{m}}(\mathbf{F}) = \begin{cases} \phi_{\mathtt{m}}(I_1) & \text{if} \quad J=1 \\ +\infty & \text{otherwise} \end{cases}$$

$\overline{W}$ **is the solution to the following optimization problem**

$$\overline{W}\left(\overline{\mathbf{F}},f_0\right) = \begin{cases} \sup\limits_{\phi_{\mathtt{m}}} \left\{ \overline{W}_0(\overline{\mathbf{F}},f_0) + (1-f_0)\min\limits_{\mathcal{J}_1}[\Psi_{\mathtt{m}}(\mathcal{J}_1)-\phi_{\mathtt{m}}(\mathcal{J}_1)] \right\} & \text{if} \quad \Psi_{\mathtt{m}}-\phi_{\mathtt{m}} > -\infty \\ \\ \inf\limits_{\phi_{\mathtt{m}}} \left\{ \overline{W}_0(\overline{\mathbf{F}},f_0) + (1-f_0)\max\limits_{\mathcal{J}_1}[\Psi_{\mathtt{m}}(\mathcal{J}_1)-\phi_{\mathtt{m}}(\mathcal{J}_1)] \right\} & \text{if} \quad \Psi_{\mathtt{m}}-\phi_{\mathtt{m}} < \infty \end{cases}$$

# From Neo-Hookean to Non-Gaussian...

**Making use of Neo-Hookean porous elastomer for the Comparison medium**

$$W_0(\mathbf{F},\mathbf{X}) = [1-\theta_0(\mathbf{X})]V_{\mathtt{m}}(\mathbf{F}) \qquad \text{with} \qquad V_{\mathtt{m}}(\mathbf{F}) = \begin{cases} \dfrac{\mu_0}{2}[I_1 - 3] & \text{if} \quad J = 1 \\ +\infty & \text{otherwise} \end{cases}$$

$$\overline{W}(\overline{\mathbf{F}}, f_0) = (1-f_0)\,\Psi_{\mathtt{m}}\left(\frac{\mathcal{I}_1}{1-f_0} + 3\right)$$

$$\mathcal{I}_1 = \frac{3(1-f_0)}{3+2f_0}\left[\overline{I}_1 - 3\right] + \frac{3}{\overline{J}^{1/3}}\left[2\overline{J} - 1 - \frac{(1-f_0)\,\overline{J}^{1/3}\left(3\overline{J}^{2/3} + 2f_0\right)}{3+2f_0} - \frac{f_0^{1/3}\overline{J}^{1/3}\,(2\overline{J} + f_0 - 2)}{(\overline{J} - 1 + f_0)^{1/3}}\right]$$

# From Neo-Hookean to Non-Gaussian...

$$\overline{W}(\overline{\mathbf{F}}, f_0) = (1 - f_0)\,\Psi_{\mathtt{m}}\left(\frac{\mathcal{I}_1}{1 - f_0} + 3\right)$$

$$\mathcal{I}_1 = \frac{3(1-f_0)}{3+2f_0}\left[\overline{I}_1 - 3\right] + \frac{3}{\overline{J}^{1/3}}\left[2\overline{J} - 1 - \frac{(1-f_0)\,\overline{J}^{1/3}\left(3\overline{J}^{2/3} + 2f_0\right)}{3+2f_0} - \frac{f_0^{1/3}\overline{J}^{1/3}\left(2\overline{J} + f_0 - 2\right)}{\left(\overline{J} - 1 + f_0\right)^{1/3}}\right]$$

**Remarks:**

- **Small deformation limit:**
  - Exact for iterative microstructure
  - Accurate approximation for arbitrary microstructures with equi-axed pores

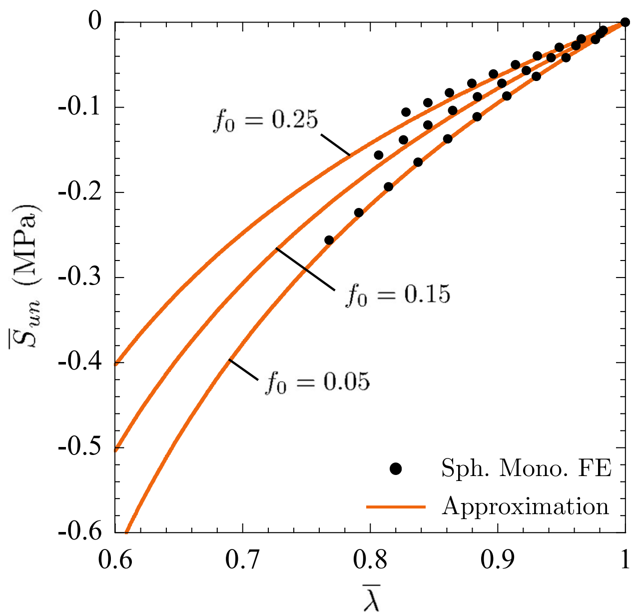
$$f = \frac{\overline{J} - 1}{\overline{J}} + \frac{f_0}{\overline{J}}$$

- **Finite deformations:**
  - Independent of $\overline{I}_2$
  - Exact evolution of the porosity
  - Accurate approximation for arbitrary loading conditions

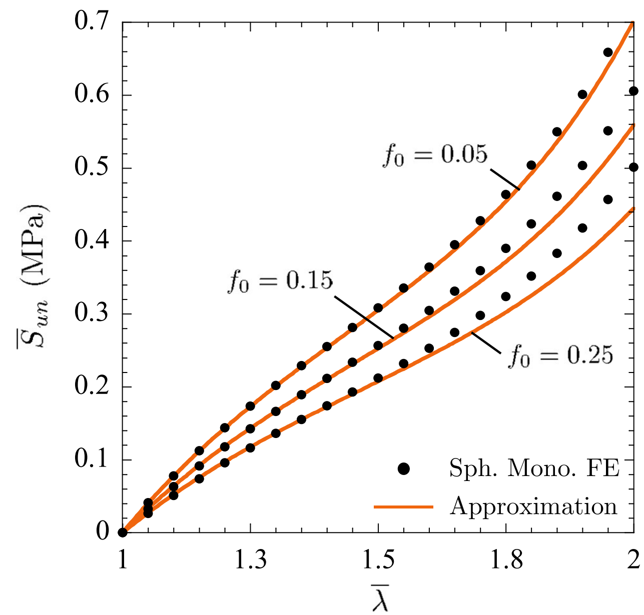
de Botton, *(2005)*

# Comparison with full field FE simulations

Uniaxial compression




Uniaxial tension



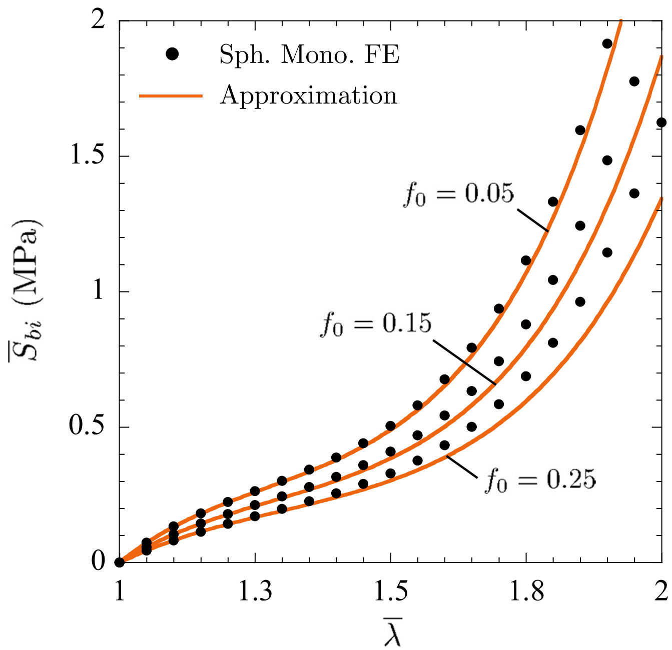

$$\overline{\lambda}_1 = \overline{\lambda},\ \overline{\lambda}_2 = \overline{\lambda}_3 = \overline{\lambda}_l$$

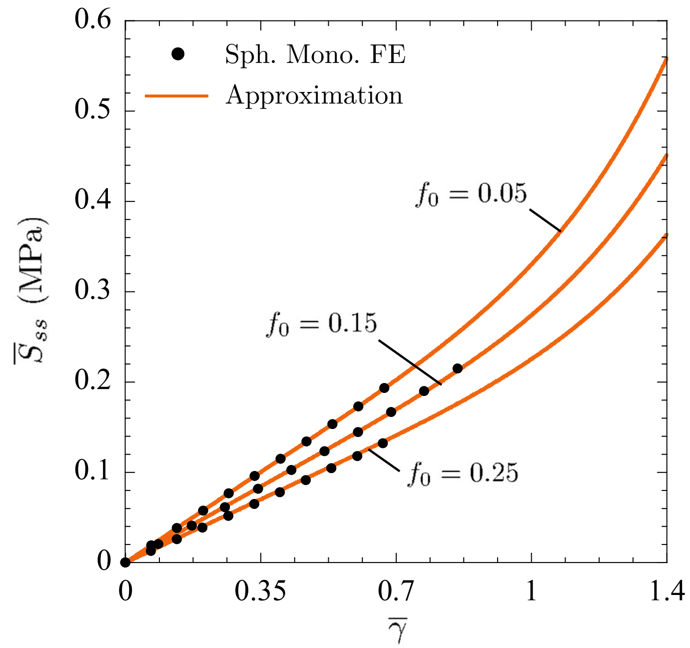
$$\overline{S}_{un} = \frac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\lambda}}(\overline{\lambda}, \overline{\lambda}_l, \overline{\lambda}_l, f_0),\ \ \frac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\lambda}_l}(\overline{\lambda}, \overline{\lambda}_l, \overline{\lambda}_l, f_0) = 0$$

# Comparison with full field FE simulations...

Biaxial tension
$$\begin{cases} \overline{\lambda}_1 = \overline{\lambda}_2 = \overline{\lambda},\ \overline{\lambda}_3 = \overline{\lambda}_l \\ \overline{S}_{bi} = \dfrac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\lambda}}(\overline{\lambda}, \overline{\lambda}, \overline{\lambda}_l, f_0),\ \dfrac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\lambda}_l}(\overline{\lambda}, \overline{\lambda}, \overline{\lambda}_l, f_0) = 0 \end{cases}$$

Simple shear
$$\begin{cases} \overline{\lambda}_1 = (\overline{\gamma} + \sqrt{\overline{\gamma}^2 + 4})/2,\ \overline{\lambda}_2 = \overline{\lambda}_1^{-1},\ \overline{\lambda}_3 = 1 \\ \overline{S}_{ss} = \dfrac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\gamma}}(\overline{\lambda}_1, \overline{\lambda}_2, 1, f_0) \end{cases}$$



# Remarks

**Closing comments:**

- **An explicit and accurate homogenization solution**
- **Readily implementable in existing FE codes (Abaqus (UHYPER), FEniCS)**

$$\boxed{\overline{W}(\overline{\mathbf{F}}, f_0) = (1 - f_0)\,\Psi_{\mathtt{m}}\left(\frac{\mathcal{I}_1}{1 - f_0} + 3\right)}$$

https://github.com/victorlefevre/UHYPER_Shrimali_Lefevre_Lopez-Pamies

**Automotive boot seal example in Abaqus**